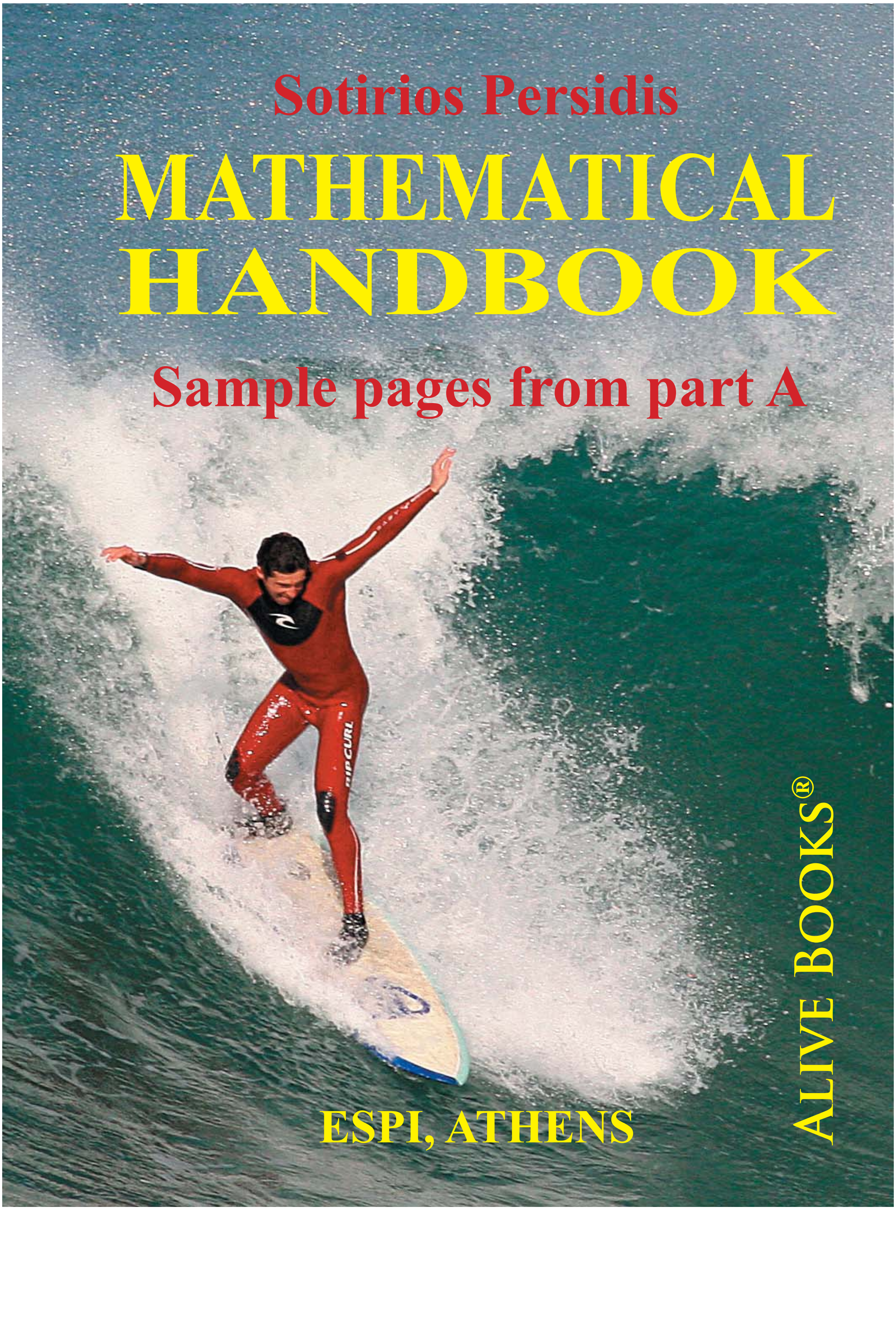
Sotirios Persidis

MATHEMATICAL HANDBOOK

Sample pages from part A



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MATHEMATICAL HANDBOOK

Sotirios Persidis
Professor Emeritus
University of Thessaloniki

This is a brief description of the first ***Alive Book***® *published, the*

MATHEMATICAL HANDBOOK.

It is a book that is alive. It consists of four parts: A, B, C, D.

Part A is a printed, high quality mathematical handbook of 340 pages in full color, very carefully designed in content, functionality and appearance. It contains more than 6000 formulas, 260 figures, and about 500 icons with connections to electronic files. **Part B** is a summary of 80 pages for fast and easy access to the essential contents of Part A. **Part C** is a collection of about 500 electronic files linked to Part A, with extensions, examples, calculations, proofs and other information available only to the readers through the Internet. **Part D** is an extensive reference to carefully selected extensions, presentations, related material, and sites available on the Internet.

The **Mathematical Handbook** is sold only as a personal copy to an individual user. For details go to **www.ifoes.org**, and click on **English** and **Mathematical Handbook**.

The underlying philosophy

This is a highly modern MATHEMATICAL HANDBOOK
consisting of two printed parts and two electronic parts that provide

- complete and clear information about almost anything needed in a daily work
- easy use for professionals and students in Mathematics, Physics, Engineering, etc.
- more information at points where it is needed, just a click away

SUMMARY OF CONTENTS

Structure and Use of the Book

Structure of the book

This is an Alive Book®. It is the first publication of a new and extended concept of a book and has four parts, A and B in printed form, C and D in electronic form:

Part A is the basic book. It contains the main material related to the cover title.

Part B is the smaller accompanying book that contains a summary of Part A.

Part C contains *additions*, i.e. files which can be obtained using the *icon numbers*.

Part D contains material in many forms related to the subject in a broader sense.

Use of the book

Each Alive Book® is the personal copy of the registered owner and can be used (in accordance with the terms stated explicitly in the registration form) as follows: Part A as a regular book (as any other handbook). Part B as a fast reference to essential material. Part C as an additional software information to more material, examples, applications, etc. Part D as a source for the best related material worldwide.

The icons used in Part A indicate where additional material is available and the type of the addition. The colors indicate the level of difficulty: Green Exa for elementary level, blue Exa for medium level, red Exa for advanced level. The three letters inside each icon indicate the content of the addition as follows:

- App — **App**lication: An application of a theory or a method.
- Cal — **Cal**culation: Calculation of an integral or an expression.
- Exa — **Exa**mple: A specific example of a case or a method.
- Ext — **Ext**ension: More theory or an extention to related material.
- Inf — **Inf**ormation: General related information.
- Pro — **Pro**of: Proof of a theorem, a statement or a formula.
- Tab — **Tab**le: Numerical table(s) of data needed in calculations.
- The — **The**ory or **The**orem: More theory or theorems or rigorous conditions.

Each icon represents an *addition* and has a naturely assigned, unique four-digit *icon number*. In part A, the following symbols are also used:

❶❷❸❹❺❻ Different cases or methods explained previously.

▶ Important points, cases or statements. ✳ Zoom of a drawing or picture.

Examples of sample pages from the book follow.

Trigonometric functions

For x in radians, the trigonometric functions $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = \cot x$ have the following graphs:

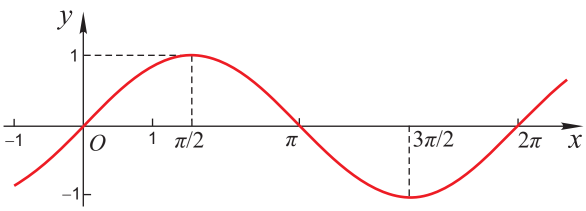

Fig. 3-3 $y = \sin x$

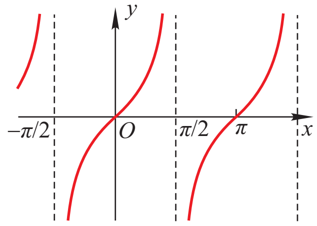

Fig. 3-5 $y = \tan x$

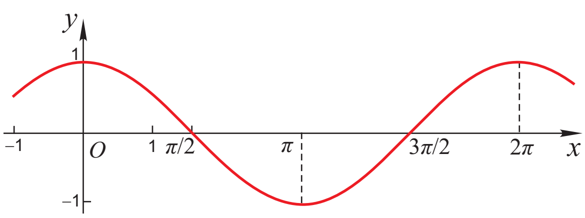

Fig. 3-4 $y = \cos x$

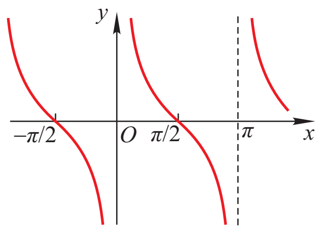

Fig. 3-6 $y = \cot x$

Values of trigonometric functions

Ext

x in degrees	x in radians	$\sin x$	$\cos x$	$\tan x$	$\cot x$
0°	0	0	1	0	$\pm\infty$
15°	$\pi/12$	$(\sqrt{6} - \sqrt{2})/4$	$(\sqrt{6} + \sqrt{2})/4$	$2 - \sqrt{3}$	$2 + \sqrt{3}$
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{3}$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	1
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$\sqrt{3}/3$
75°	$5\pi/12$	$(\sqrt{6} + \sqrt{2})/4$	$(\sqrt{6} - \sqrt{2})/4$	$2 + \sqrt{3}$	$2 - \sqrt{3}$
90°	$\pi/2$	1	0	$\pm\infty$	0

For angles in other quadrants, we can use the transformation formulas of Section 3.2.

4 GEOMETRY

4.1 Plane Geometry

Ext

Triangle

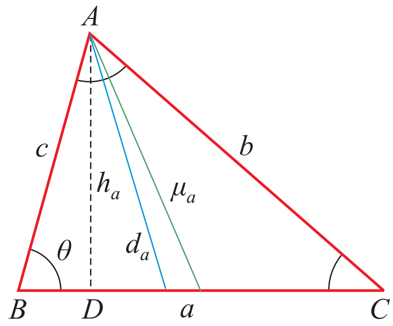

Fig. 4-1

Side $b^2 = a^2 + c^2 \pm 2a(BD)$
$(+ \text{ if } \theta > 90°, - \text{ if } \theta < 90°)$

Perimeter $\Pi = 2s = a + b + c$

Height $h_a = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$

Median $\mu_a{}^2 = \frac{1}{2}(b^2 + c^2) - \frac{1}{4}a^2$

Bisector (inner) $d_a = \frac{2\sqrt{bcs(s-a)}}{b+c}$

Bisector (outer) $D_a = \frac{2\sqrt{bc(s-b)(s-c)}}{|b-c|}, \quad b \neq c$

Area $E = \frac{1}{2}ah_a = \frac{1}{2}ac\sin\theta = \sqrt{s(s-a)(s-b)(s-c)}$

Radius of inscribed circle $\rho = \frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)} = \left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)^{-1}$

Radius of circumscribed circle $R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$

Right triangle For $\theta = 90°$, $b^2 = a^2 + c^2$ (Pythagorean theorem)

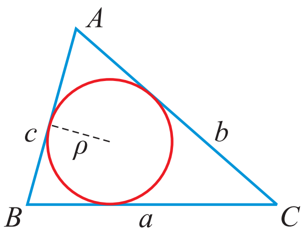

Fig. 4-2

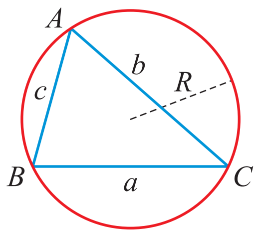

Fig. 4-3

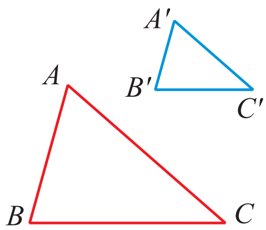

Fig. 4-4

Similar triangles (Fig. 4-4)

$A = A' \quad B = B' \quad C = C'$ $\qquad \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$

With tan *ax*

❶ Set $u = \sin ax$ or $u = \cos ax$.
❷ Set $u = \tan ax$.
❸ Write integrand as sum.

$$\int \tan ax\,dx = -\frac{1}{a}\ln|\cos ax|$$ ❶

$$\int \tan^2 ax\,dx = \frac{\tan ax}{a} - x$$ ❸

$$\int \tan^3 ax\,dx = \frac{\tan^2 ax}{2a} + \frac{1}{a}\ln|\cos ax|$$ ❸

$$\int \frac{dx}{\tan ax} = \frac{1}{a}\ln|\sin ax|$$ ❶

$$\int \frac{dx}{\tan ax\cos^2 ax} = \frac{1}{a}\ln|\tan ax|$$ ❷

$$\int x\tan ax\,dx = \sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)|B_{2k}|a^{2k-1}x^{2k+1}}{(2k+1)!} = \frac{ax^3}{3} + \frac{a^3x^5}{15} + \frac{2a^5x^7}{105} + \cdots,\quad |x| < \frac{\pi}{2}$$

$$\int \frac{\tan ax}{x}\,dx = \sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)|B_{2k}|a^{2k-1}x^{2k-1}}{(2k-1)(2k)!} = ax + \frac{a^3x^3}{9} + \frac{2a^5x^5}{75} + \cdots,\quad |x| < \frac{\pi}{2}$$

Cal

$$\int x\tan^2 ax\,dx = \frac{x\tan ax}{a} + \frac{1}{a^2}\ln|\cos ax| - \frac{x^2}{2}$$ ❸

$$\int \frac{dx}{b\tan ax + c} = \int \frac{\cos ax\,dx}{b\sin ax + c\cos ax} = \frac{1}{b^2+c^2}\left[cx + \frac{b}{a}\ln|b\sin ax + c\cos ax|\right]$$

$$\int \frac{\tan ax\,dx}{b\tan ax + c} = \int \frac{\sin ax\,dx}{b\sin ax + c\cos ax} = \frac{1}{b^2+c^2}\left[bx - \frac{c}{a}\ln|b\sin ax + c\cos ax|\right]$$

Cal

$$\int \frac{dx}{b\tan^2 x + c} = \frac{1}{c-b}\left[x - \sqrt{\frac{b}{c}}\tan^{-1}\left(\sqrt{\frac{b}{c}}\tan x\right)\right],\quad b \neq c,\ bc > 0$$ ❷

$$\int \frac{\tan x\,dx}{1 + k^2\tan^2 x} = \frac{1}{2(k^2-1)}\ln(\cos^2 x + k^2\sin^2 x)$$ ❶

$$\int \tan^n ax\,dx = \frac{\tan^{n-1} ax}{(n-1)a} - \int \tan^{n-2} ax\,dx,\quad n \neq 1$$ ❸

With logarithmic functions

❶ Use indefinite integral.
❷ Use complex integral.
❸ Set $x = e^{-y}$.
❹ Set $x = \exp(-z^2)$, $z > 0$.
❺ Use gamma function.

$$\int_0^1 \frac{\ln x}{1 \pm x}\,dx = (-3 \pm 1)\frac{\pi^2}{24}$$ (❸, then ❷) Cal

$$\int_0^1 \frac{\ln x}{(1+x)^2}\,dx = -\ln 2$$ ❶

$$\int_0^1 \frac{\ln x}{1+x^2}\,dx = -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = -G = -0.915965594177\ldots$$ Cal

$$\int_0^1 \frac{\ln x}{\sqrt{1-x^2}}\,dx = -\frac{\pi}{2}\ln 2$$ (set $x = \cos y$, then ❷)

$$\int_0^1 \frac{x \ln x}{\sqrt{1-x^2}}\,dx = \ln 2 - 1$$ (set $x = \cos y$, then ❷)

$$\int_0^1 (\ln x)\sqrt{1-x^2}\,dx = -\frac{\pi}{8}(1 + 2\ln 2)$$ (set $x = \cos y$, then ❷)

$$\int_0^1 x(\ln x)\sqrt{1-x^2}\,dx = \frac{1}{3}\ln 2 - \frac{4}{9}$$ (set $x = \cos y$, then ❷)

Cal (for the four integrals above)

$$\int_0^1 \sqrt{|\ln x|}\,dx = \frac{\sqrt{\pi}}{2}$$ ❹

$$\int_0^1 \frac{dx}{\sqrt{|\ln x|}} = \sqrt{\pi}$$ ❹

$$\int_0^1 \frac{x^{p-1}}{\sqrt{|\ln x|}}\,dx = \sqrt{\frac{\pi}{p}}, \quad p > 0$$ ❹

$$\int_0^1 (\ln x)\ln(1+x)\,dx = 2 - 2\ln 2 - \frac{\pi^2}{12}$$

$$\int_0^1 (\ln x)\ln(1-x)\,dx = 2 - \frac{\pi^2}{6}$$

Cal (for the two integrals above)

$$\int_0^1 x^{p-1}\ln|\ln x|\,dx = \int_0^\infty e^{-px}\ln x\,dx = -\frac{1}{p}(\gamma + \ln p), \quad p > 0$$ ❸❺ [γ = Euler constant]

$$\ln|\sin x| = \ln|x| - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \cdots = \ln|x| - \sum_{k=1}^{\infty} \frac{2^{2k-1}|B_{2k}|}{k(2k)!} x^{2k}, \quad 0 < |x| < \pi$$

$$\ln|\cos x| = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \cdots = -\sum_{k=1}^{\infty} \frac{2^{2k-1}(2^{2k}-1)|B_{2k}|}{k(2k)!} x^{2k}, \quad |x| < \frac{\pi}{2}$$

Cal

$$\ln|\tan x| = \ln|x| + \frac{x^2}{3} + \frac{7x^4}{90} + \frac{62x^6}{2835} + \cdots = \ln|x| + \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k-1}-1)|B_{2k}|}{k(2k)!} x^{2k}, \quad 0 < |x| < \frac{\pi}{2}$$

Hyperbolic functions

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \qquad -\infty < x < \infty$$

$$\frac{1}{\sinh x} = \frac{1}{x} - \frac{x}{6} + \frac{7x^3}{360} - \frac{31x^5}{15120} + \cdots = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2(2^{2k-1}-1)B_{2k}}{(2k)!} x^{2k-1}, \quad 0 < |x| < \pi$$ Cal

$$\frac{1}{\sinh x} = \frac{1}{x} - 2x\left[\frac{1}{x^2+\pi^2} - \frac{1}{x^2+2^2\pi^2} + \frac{1}{x^2+3^2\pi^2} - \cdots\right] = \frac{1}{x} + 2x\sum_{k=1}^{\infty} \frac{(-1)^k}{x^2+k^2\pi^2}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \qquad -\infty < x < \infty$$

$$\frac{1}{\cosh x} = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \cdots = \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} x^{2k}, \qquad |x| < \frac{\pi}{2}$$ Cal

$$\frac{1}{\cosh x} = 4\pi\left[\frac{1}{4x^2+\pi^2} - \frac{3}{4x^2+3^2\pi^2} + \frac{5}{4x^2+5^2\pi^2} - \cdots\right] = 4\pi\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k-1)}{4x^2+(2k-1)^2\pi^2}$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!} x^{2k-1}, \qquad |x| < \frac{\pi}{2}$$ Cal

$$\tanh x = 8x\left[\frac{1}{4x^2+\pi^2} + \frac{1}{4x^2+3^2\pi^2} + \frac{1}{4x^2+5^2\pi^2} + \cdots\right] = 8x\sum_{k=1}^{\infty} \frac{1}{4x^2+(2k-1)^2\pi^2}$$

$$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \cdots = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2^{2k}B_{2k}}{(2k)!} x^{2k-1}, \qquad 0 < |x| < \pi$$ Cal

$$\coth x = \frac{1}{x} + 2x\left[\frac{1}{x^2+\pi^2} + \frac{1}{x^2+2^2\pi^2} + \frac{1}{x^2+3^2\pi^2} + \cdots\right] = \frac{1}{x} + 2x\sum_{k=1}^{\infty} \frac{1}{x^2+k^2\pi^2}$$

11.4 Tables of Fourier series

In each case below, the following are given: the function $f(x)$ with the interval $I = (c, c+2L)$, the Fourier series $F(x)$, the discontinuities x_d of $f(x)$, and the values $F(x_d)$, the graphs of $f(x)$ in red and $F_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L}\right)$ in green.

Odd functions

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

$$F(x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{2k-1}$$

$x_d = n\pi, \; F(x_d) = 0$

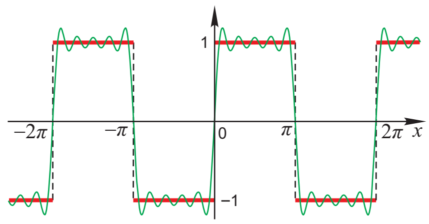

Fig. 11-1: $f(x)$ —, $F_{10}(x)$ —

$f(x) = x, \quad -\pi < x < \pi$

$$F(x) = 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\sin kx}{k}$$

Cal

$x_d = (2n+1)\pi, \; F(x_d) = 0$

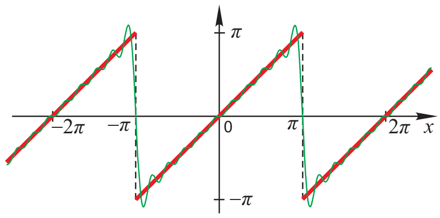

Fig. 11-2: $f(x)$ —, $F_{10}(x)$ —

$$f(x) = \begin{cases} -(\pi + x)/2, & -\pi \le x < 0 \\ (\pi - x)/2, & 0 < x \le \pi \end{cases}$$

$$F(x) = \sum_{k=1}^{\infty}\frac{\sin kx}{k}$$

Cal

$x_d = 2n\pi, \; F(x_d) = 0$

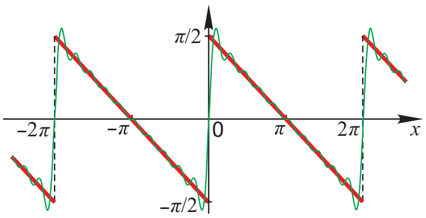

Fig. 11-3: $f(x)$ —, $F_{10}(x)$ —

$$f(x) = \begin{cases} -x^2, & -\pi < x < 0 \\ x^2, & 0 \le x < \pi \end{cases}$$

$$F(x) = \sum_{k=1}^{\infty}\frac{2k^2\pi^2(-1)^{k+1} + 4(-1)^k - 4}{k^3\pi}\sin kx$$

$x_d = (2n+1)\pi, \; F(x_d) = 0$

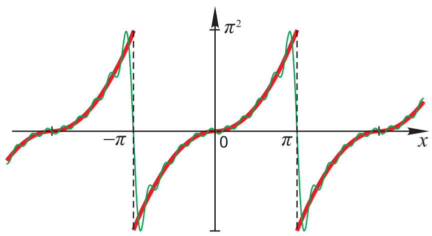

Fig. 11-4: $f(x)$ —, $F_{10}(x)$ —

We define the *surface integral* of **A** on S (S' is the projection of S on xOy)

$$\int_S \mathbf{A}\cdot d\mathbf{S} = \iint_{S'} \mathbf{A}\cdot\mathbf{N}\frac{dx\,dy}{|\mathbf{N}\cdot\mathbf{k}|} = \lim_{n\to\infty}\sum_{k=1}^{n}\mathbf{A}_k(x_k, y_k, z_k)\cdot\mathbf{N}_k\Delta E_k$$

Also, $\int_S \mathbf{A}\times d\mathbf{S} = \lim_{n\to\infty}\sum_{k=1}^{n}\mathbf{A}_k(x_k, y_k, z_k)\times\mathbf{N}_k\Delta E_k$ is defined in a similar way.

Theorems of Gauss, Stokes and Green

In a three dimensional Euclidean space, let S be a piecewise smooth oriented closed surface (Fig. 12-6), which encloses a bounded, simply connected region V. Let **N** be the unit vector normal to S toward the outside and $d\mathbf{S} = \mathbf{N}dS$. Then, according to *Gauss's theorem*, for a vector field **A** with continuous partial derivatives, we have

$$\int_V \nabla\cdot\mathbf{A}\,dV = \int_S \mathbf{A}\cdot d\mathbf{S}$$

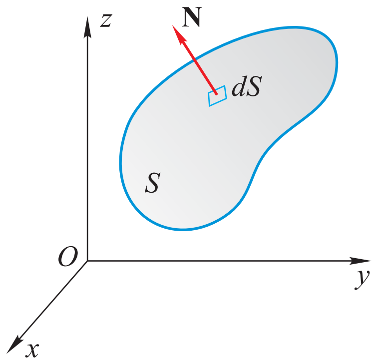

Fig. 12-6

Let S be a piecewise smooth oriented open surface whose boundary is a piecewise smooth simple closed curve C (Fig. 12-7), and $d\mathbf{S} = \mathbf{N}dS$ (**N** the unit vector normal to S). Then according to *Stokes's theorem*, for a vector field **A** with continuous partial derivatives we have

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \int_S (\nabla\times\mathbf{A})\cdot d\mathbf{S}$$

where the line integral on the closed curve C has been obtained with the appropriate direction (someone walking on S, on the side of **N** and close to C, has the inside of S at his left).

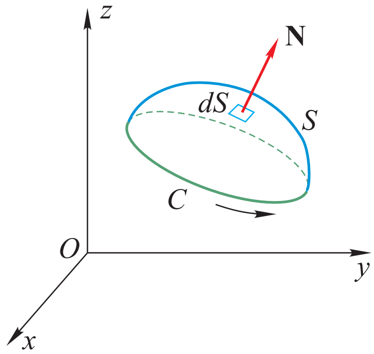

Fig. 12-7

If D is a domain of the xy plane containing a piecewise smooth and simple closed curve C and its interior R, then according to *Green's theorem in the plane*, we have

$$\oint_C \mathbf{A}\cdot\mathbf{T}\,ds = \oint_C (P\,dx + Q\,dy) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$ **Ext**

This can be obtained from Stokes's theorem with $\mathbf{A} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ and C a closed curve in the xy plane.

Oblate spheroidal coordinates (ξ, η, φ)

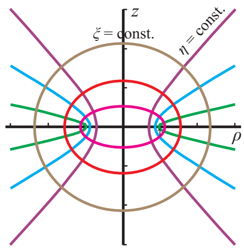

Fig. 13-7

$$x = a\cosh\xi\cos\eta\cos\varphi,\quad y = a\cosh\xi\cos\eta\sin\varphi,$$
$$z = a\sinh\xi\sin\eta$$

with $0 \le \xi < \infty,\ -\pi/2 \le \eta \le \pi/2,\ 0 \le \varphi < 2\pi$

$$h_1 = h_\xi = h_2 = h_\eta = a\sqrt{\sinh^2\xi + \sin^2\eta}\,,$$
$$h_3 = a\cosh\xi\cos\eta$$

Setting $w^2 = a^2(\sinh^2\xi + \sin^2\eta)$ we have

$$\nabla^2\Phi = \frac{1}{w^2\cosh\xi}\frac{\partial}{\partial\xi}\left(\cosh\xi\frac{\partial\Phi}{\partial\xi}\right) + \frac{1}{w^2\cos\eta}\frac{\partial}{\partial\eta}\left(\cos\eta\frac{\partial\Phi}{\partial\eta}\right) + \frac{1}{a^2\cosh^2\xi\cos^2\eta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

Two families of coordinate surfaces result from the rotation of Fig. 13-6 around its y axis, which then becomes z. The third family of coordinate surfaces consists of planes that include this axis. In a plane that includes the new z axis, the coordinate curves (Fig. 13-7) are given for various values of ξ and η by the equations

$$\frac{\rho^2}{\cosh^2\xi} + \frac{z^2}{\sinh^2\xi} = a^2 \quad\text{and}\quad \frac{\rho^2}{\cos^2\eta} - \frac{z^2}{\sin^2\eta} = a^2 \quad \text{where } \rho = (x^2 + y^2)^{1/2}$$

Prolate spheroidal coordinates (ξ, η, φ)

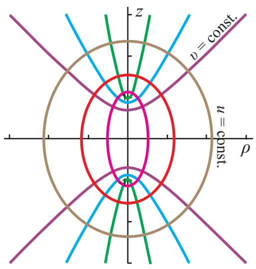

Fig. 13-8

$$x = a\sinh\xi\sin\eta\cos\varphi,\quad y = a\sinh\xi\sin\eta\sin\varphi,$$
$$z = a\cosh\xi\cos\eta$$

with $0 \le \xi < \infty,\ \ 0 \le \eta \le \pi,\ \ 0 \le \varphi < 2\pi$

$$h_1 = h_\xi = h_2 = h_\eta = a\sqrt{\sinh^2\xi + \sin^2\eta}\,,$$
$$h_3 = h_\varphi = a\sinh\xi\sin\eta$$

Setting $w^2 = a^2(\sinh^2\xi + \sin^2\eta)$ we have

$$\nabla^2\Phi = \frac{1}{w^2}\frac{\partial}{\partial\xi}\left(\sinh\xi\frac{\partial\Phi}{\partial\xi}\right) + \frac{1}{w^2\sin\eta}\frac{\partial}{\partial\eta}\left(\sin\eta\frac{\partial\Phi}{\partial\eta}\right) + \frac{1}{a^2\sinh^2\xi\sin^2\eta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

Two families of coordinate surfaces result from the rotation of Fig. 13-6 around its x axis, which then becomes z. The third family of coordinate surfaces consists of planes that include this axis. In a plane that includes the new z axis, the coordinate curves (Fig. 13-8) are given for various values of ξ and η by the equations

$$\frac{\rho^2}{\sinh^2\xi} + \frac{z^2}{\cosh^2\xi} = a^2 \quad\text{and}\quad \frac{z^2}{\cos^2\eta} - \frac{\rho^2}{\sin^2\eta} = a^2 \quad \text{where } \rho = (x^2 + y^2)^{1/2}$$

14 BESSEL FUNCTIONS

14.1 Definitions

The functions that satisfy *Bessel's differential equation*

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

Ext

are the *Bessel functions of order n*.

The general solution of Bessel's differential equation is

$$y = c_1J_n(x) + c_2J_{-n}(x), \qquad n \neq 0, 1, 2, \ldots$$

$$y = c_1J_n(x) + c_2Y_n(x), \qquad \text{for any } n$$

$$y = c_1H_n^{(1)}(x) + c_2H_n^{(2)}(x), \qquad \text{for any } n$$

$$y = c_1J_n(x) + c_2J_n(x)\int \frac{dx}{xJ_n^2(x)} \qquad \text{for any } n$$

where c_1 and c_2 are arbitrary constants and $J_n(x)$, $Y_n(x)$ are the Bessel functions of the first and second kind, respectively.

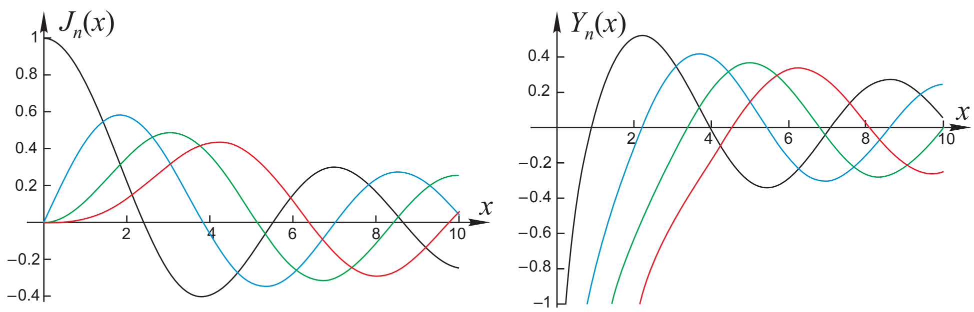

Fig. 14-1: $J_n(x)$, $Y_n(x)$, $n = 0$ ——, $n = 1$ ——, $n = 2$ ——, $n = 3$ ——

14.2 Bessel Functions of the First Kind

The Bessel functions of the first kind and order n are defined by the relations

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} - \cdots\right\}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+n}}{k!\,\Gamma(k+n+1)}$$

Values

$$T_n(-x) = (-1)^n T_n(x)$$

$$T_{2n}(0) = (-1)^n \qquad T_{2n+1}(0) = 0 \qquad T_n(1) = 1 \qquad T_n(-1) = (-1)^n$$

Expansion in series

$$f(x) = \tfrac{1}{2}a_0 T_0(x) + a_1 T_1(x) + a_2 T_2(x) + \cdots$$

Exa

$$a_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}\,dx$$

16.8 Chebyshev Polynomials of the Second Kind

Differential equation

The polynomials $U_n(x)$ satisfy the differential equation

$$(1 - x^2)y'' - 3xy' + n(n + 2)y = 0$$

Generating function

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n, \quad |t|<1$$

First polynomials

$U_0(x) = 1$

$U_1(x) = 2x$

$U_2(x) = 4x^2 - 1$

$U_3(x) = 8x^3 - 4x$

$U_4(x) = 16x^4 - 12x^2 + 1$

$U_5(x) = 32x^5 - 32x^3 + 6x$

$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$

$U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x$

$U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1$

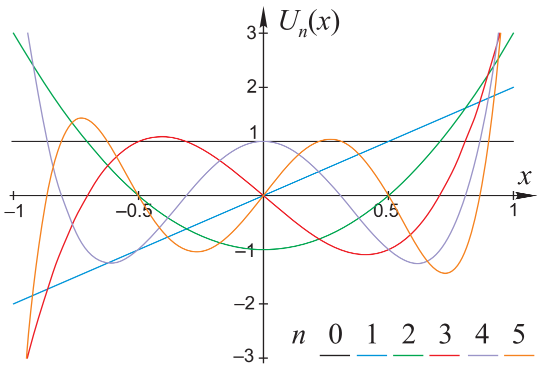

Fig. 16-5

$$U_n(x) = \binom{n+1}{1}x^n - \binom{n+1}{3}x^{n-2}(1-x^2) + \cdots = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(n-k)!}{k!(n-2k)!}(2x)^{n-2k}, \quad n \ge 1$$

$$U_n(x) = \frac{(-1)^n 2^n (n+1)!}{(2n+1)!\sqrt{1-x^2}} \frac{d^n}{dx^n}\left[\sqrt{1-x^2}\,(1-x^2)^n\right] \quad \text{(Rodrigues's formula)}$$

18.4 Tables of Fourier Transforms

In each case, we give (a) the function $f(t)$, (b) the corresponding Fourier transform $F(\omega)$ [or $F_s(\omega)$ or $F_c(\omega)$], (c) the graph of $f(t)$ in green, (d) the graph of $\text{Re}\{F(\omega)\}$ in red, and (e) the graph of $\text{Im}\{F(\omega)\}$ in purple. On the horizontal axis, the values of t and ω are given and on the vertical axis the values of $f(t)$ and $F(\omega)$ are given. For some $f(t)$ the integral $\int_{-\infty}^{\infty}|f(t)|\,dt$ does not exist, but the function $F(\omega)$ can be used in formal (not rigorous) calculations.

Methods to prove the formulas: ❶ Use delta function. ❷ Use definite integral. ❸ Use Fourier cosine integral. ❹ Use complex integral. ❺ Prove inverse.

Fourier transforms (— $f(t)$, — $\text{Re}\{F(\omega)\}$, — $\text{Im}\{F(\omega)\}$, $-\infty < \omega < \infty$) **Inf**

$f(t) = 1$ ❶

$$F(\omega) = \sqrt{2\pi}\,\delta(\omega)$$

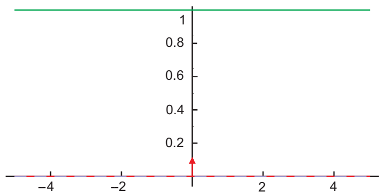

Fig. 18-1

$f(t) = \delta(t-a)$ ❶

$$F(\omega) = \frac{1}{\sqrt{2\pi}}e^{ia\omega}$$

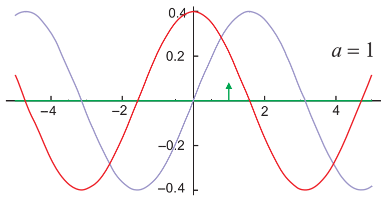

Fig. 18-2

$$f(t) = \begin{Bmatrix} 1, & |t| < a \\ 0, & |t| > a \end{Bmatrix}, \quad a > 0$$

Cal

$$F(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{\sin a\omega}{\omega}$$

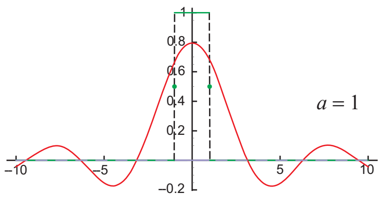

Fig. 18-3

$$f(t) = \begin{Bmatrix} 1 - |t|/a, & |t| < a \\ 0, & |t| > a \end{Bmatrix}, \quad a > 0$$

Cal

$$F(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{1 - \cos a\omega}{a\omega^2}$$

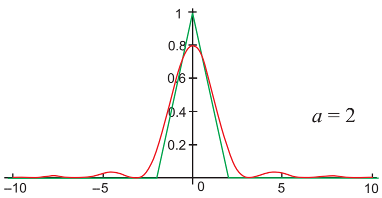

Fig. 18-4

19.3 Tables of Laplace Transforms

Laplace transforms of some elementary functions Pro

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\frac{1}{s}, \quad s > 0$	$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
t	$\frac{1}{s^2}, \quad s > 0$	$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
$t^n, \quad n = 0, 1, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$	$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \|a\|$
$t^a, \quad a > -1$	$\frac{\Gamma(a+1)}{s^{a+1}}, \quad s > 0$	$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \|a\|$
e^{at}	$\frac{1}{s-a}, \quad s > a$	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$t^n e^{at}, \quad n = 0, 1, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$t^b e^{at}, \quad b > -1$	$\frac{\Gamma(b+1)}{(s-a)^{b+1}}$	$U_a(t) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$	$\frac{e^{-as}}{s}, \quad s > 0, a \ge 0$
$\ln t$	$-\frac{\gamma + \ln s}{s}, \quad s > 0$	$\delta(t-a), \quad a \ge 0$ [delta function]	e^{-as}

Inverse Laplace transforms

$F(s)$	$f(t)$	$F(s)$	$f(t)$
s^{-1}	1	$(s^2+a^2)^{-1}$	$a^{-1}\sin at$
$s^{-n}, \quad n = 1, 2, \ldots$	$t^{n-1}/(n-1)!$	$s(s^2+a^2)^{-1}$	$\cos at$
$s^{-a}, \quad a > 0$	$t^{a-1}/\Gamma(a)$	$(s^2-a^2)^{-1}$	$a^{-1}\sinh at$
$(s-a)^{-1}$	e^{at}	$s(s^2-a^2)^{-1}$	$\cosh at$
$(s-a)^{-n}, \quad n = 1, 2, \ldots$	$t^{n-1}e^{at}/(n-1)!$	$[(s-a)^2+b^2]^{-1}$	$b^{-1}e^{at}\sin bt$
$(s-a)^{-b}, \quad b > 0$	$t^{b-1}e^{at}/\Gamma(b)$	$s[(s-a)^2+b^2]^{-1}$	$b^{-1}e^{at}(b\cos bt + a\sin bt)$

21.3 Various Distributions

Normal distribution

The continuous random variable X has the probability (i.e. density) function

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

for $-\infty < x < \infty$ and distribution function

$$F(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} \exp\left[-\frac{(u-\mu)^2}{2\sigma^2}\right] du$$

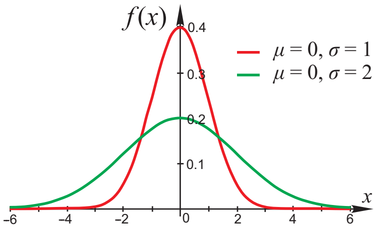

Fig. 21-1

Mean value μ, variance σ^2, skewness $a_3 = 0$,
kurtosis $a_4 = 3$, moment generating function $M(t) = \exp(\mu t + \sigma^2 t^2/2)$,
characteristic function $\Phi(\omega) = \exp(i\mu\omega - \sigma^2\omega^2/2)$

Setting $Z = (X-\mu)/\sigma$ we obtain the *standard normal distribution* with probability function and distribution function respectively

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \qquad F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right]$$

Tab

Binomial distribution

Let p be the probability for an event to happen (success) in performing a random experiment once (single trial) and $q = 1 - p$ be the probability for the same event not to happen (failure). If we repeat the experiment n times, then the probability for this event to happen exactly x times ($x = 0, 1, \ldots, n$) is given by the *binomial distribution*

$$f(x) = P(X = x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

i.e. the coefficients of the binomial expansion

$$(p+q)^n = q^n + \binom{n}{1} p q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \cdots + p^n$$

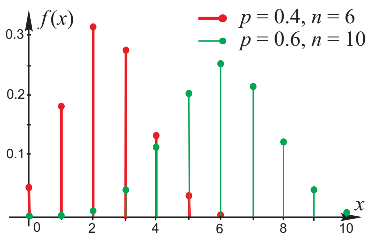

Fig. 21-2

Mean value $\mu = np$, variance $\sigma^2 = npq$,

skewness $a_3 = \dfrac{q-p}{\sqrt{npq}}$, kurtosis $a_4 = \dfrac{3(n-2)pq+1}{npq}$

moment generating function $M(t) = (pe^t + q)^n$,
characteristic function $\Phi(\omega) = (pe^{i\omega} + q)^n$.

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